thmTheorem Lemma

Sum-of-squares decompositions for a family of noncontextuality inequalities and self-testing of quantum devices

Published in Quantum 4, 302 (2020)

Debashis Saha, Rafael Santos, and Remigiusz Augusiak

QTech - Nov 2020

Center for Theoretical Physics - Polish Academy of Sciences

Brief Introduction

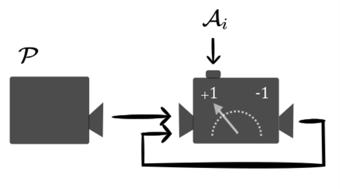
Brief Introduction

- Violation of a noncontextuality inequality or the phenomenon referred to 'quantum contextuality' is a fundamental feature of quantum theory.
- It is a fundamental problem to understand what is the maximal information about the underlying quantum system that can be inferred from the correlations observed in a contextuality experiment.
- And the main purpose of self-testing is exploit the information can be used for certification of quantum devices from minimal assumptions of their internal functioning.

Sequential-measurement set-up

Sequential-measurement set-up

Each run of the experimental observation comprises of preparation of a physical system followed by two measurements in a sequence using one non-demolishing measurement device.



The measurement device has n (odd) different settings, each of which yields ± 1 outcome.

Sequential-measurement set-up

Assumption

The measurement device only returns the actual post-measurement state.

In quantum theory, a measurement will be represented by the following operator

$$A_i = 2F_i - 1 \tag{1}$$

where $F_i \leq 1$.

Assumption

The measurements are realized in a particular way such that F_i are projectors.

Modified KCBS-inequality with SOS decompositions

The inequality for $n = 2^m + 1 (m \geq 2)$

The linear expression considered to test nonclassicality (or noncontextuality in the usual scenario) in this set-up is given by,

$$\mathcal{B}_n := -\frac{1}{2}\sum_i (\langle \mathcal{A}_i \mathcal{A}_{i+1} \rangle + \langle \mathcal{A}_{i+1} \mathcal{A}_i \rangle) - \gamma \sum_i \langle \mathcal{A}_i \rangle \leqslant \eta_n^C \tag{2}$$

where γ is a real coefficient that depends of n and

$$\eta_n^C < \eta_n^Q = \sup_{|\psi\rangle, A_i} \langle \psi | B_n | \psi \rangle \tag{3}$$

where B_n is the associate quantum operator.

The SOS decompositions are designed to be:

$$B_n = \eta_n^Q 1 - \sum_k E_k^\dagger E_k \tag{4}$$

where E_k is given explicitly as function of the measurement operators A_i.

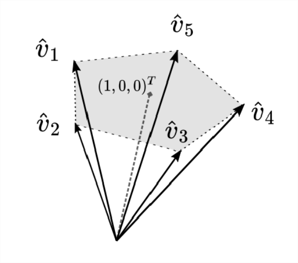
Simplest case ($n = 5$)

$$\mathcal{B} = -\frac{1}{2}\sum_{i=1}^{5}(\langle \mathcal{A}_i \mathcal{A}_{i+1}\rangle + \langle \mathcal{A}_{i+1}\mathcal{A}_i\rangle) - \alpha^2 \sum_{i=1}^{5}\langle \mathcal{A}_i\rangle \leq \eta^C = 3 + \alpha^2 \qquad (5)$$

where $\alpha = \frac{1}{2}\sec \pi/5$.

Result (Modified KCBS inequality with SOS)
The maximal quantum value of $\mathcal{B}$ given in Eq. (5) *is* $\eta^Q = 3(1+\alpha^2)$.

A particular quantum realization achieving the optimal quantum realization in a 3-dim Hilbert space is given by the state $|\widehat{\psi}\rangle = |0\rangle \equiv (1,0,0)^T$ and observables $\widehat{A}_i = 2|\widehat{v}_i\rangle\langle\widehat{v}_i| - 1$, where $|\widehat{v}_i\rangle$ are three-dimensional real vectors that can be geometrically represented in the following figure:



Self-testing

Definition *(Self-testing of preparation $|\bar{\psi}\rangle \in C^d$ and a set of measurements $\{\bar{A}_i\}_{i=1}^n$ acting on C^d).* If a set of observables $\{A_i\}_{i=1}^n$ acting on unknown finite-dimensional Hilbert space $\mathcal{H}$ and a state $|\psi\rangle \in \mathcal{H}$ maximally violate a noncontextuality inequality, then there exists an isometry $\Phi : \mathcal{H} \mapsto C^d$ such that

1. $\Phi|\psi\rangle = |\bar{\psi}\rangle$,
2. $\Phi A_i \Phi^\dagger = \bar{A}_i$ for all $i = 1, \ldots, n$.

Result (Self-testing)
Under the Assumptions 1 and 2, if a quantum state $|\psi\rangle \in \mathcal{H}$ and a set of n (where $n = 2^m + 1, m \in N \setminus \{1\}$) measurements A_i acting on $\mathcal{H}$ violate the inequality (2) maximally, then there exists a projection $P : \mathcal{H} \to C^3$ and a unitary U acting on C^3 such that

$$U(PA_iP^\dagger)U^\dagger = 2|\widehat{v}_i\rangle\langle\widehat{v}_i| - 1_3,$$
$$U(P|\psi\rangle) = (1, 0, 0)^T, \tag{6}$$

where $|\widehat{v}_i\rangle$ are defined in (7).

$$|\widehat{v}_i\rangle = (\cos(\theta), \sin(\theta)\sin(\varphi_i), \sin(\theta)\cos(\varphi_i))^T,$$
$$\cos(\theta) = \sqrt{\frac{1}{1+2\alpha^{(n)}}},\ \varphi_i = \frac{(n-1)\pi}{n} i. \tag{7}$$

The main ideia behind the proof is exploit the algebraic relations which came from the 'saturation of the squares' in the Sum-of-squares decompositions.