Mapping an infinite and a finite Hadamard quantum walk onto a unique case of a random walk process

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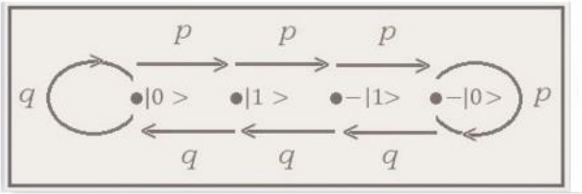
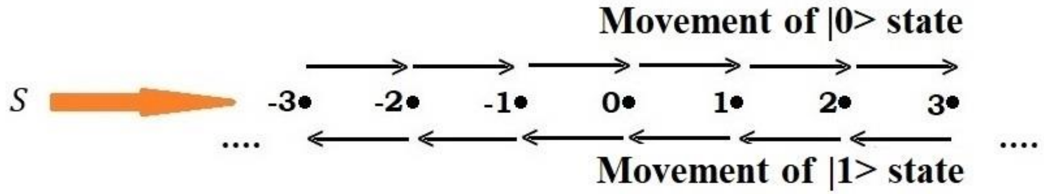
1. Mapping a Hadamard operator to a four-site symmetric Markov chain

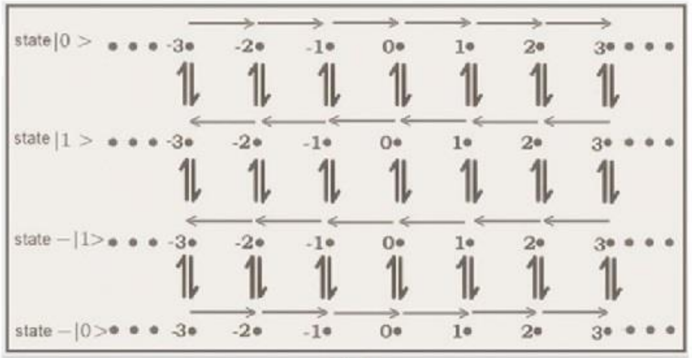
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



The shift operator is a birth-death process



2. Combining a four-site symmetric Markov chain with the shift operators yields:



Inserting boundaries into the RW model

1. Mapping a Hadamard operator to a four-site symmetric Markov chain

The Hadamard matrix, H, is defined by the following unitary operator:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad (1)$$

This operator can be mapped by four states of a Markov chain, as shown in the following figure, with transition probabilities of $p = q = 0.5$.

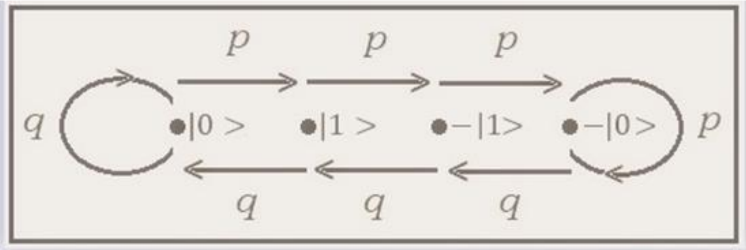


Figure 1. A Markov chain represents the transitions between four states. Note that each site is named with the following symbols respectively:

$|0>, |1>, -|1>, -|0>$

This can be shown explicitly by the following product:

$$H = \frac{1}{\sqrt{2}} BAB^T \qquad (2)$$

where A is the transition probability matrix of the system described in figure 1:

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

and matrix B is defined as:

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

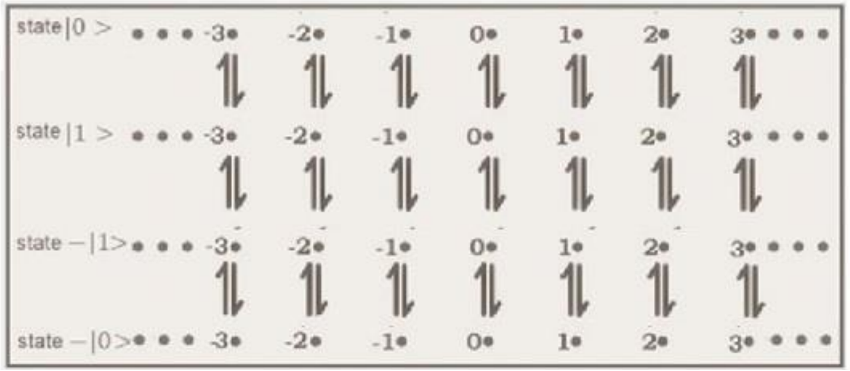
which will be called in this context:

'an interference matrix.'

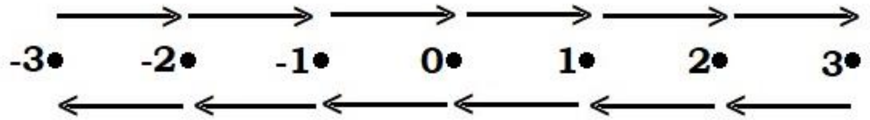
Note that $\text{Det}(H) \neq 0$, while $\text{Det}(A) = 0$.

2. Combining the symmetric Markov chain with the shift operator yields a RW model

$I_d \otimes A \rightarrow$



$S \rightarrow$

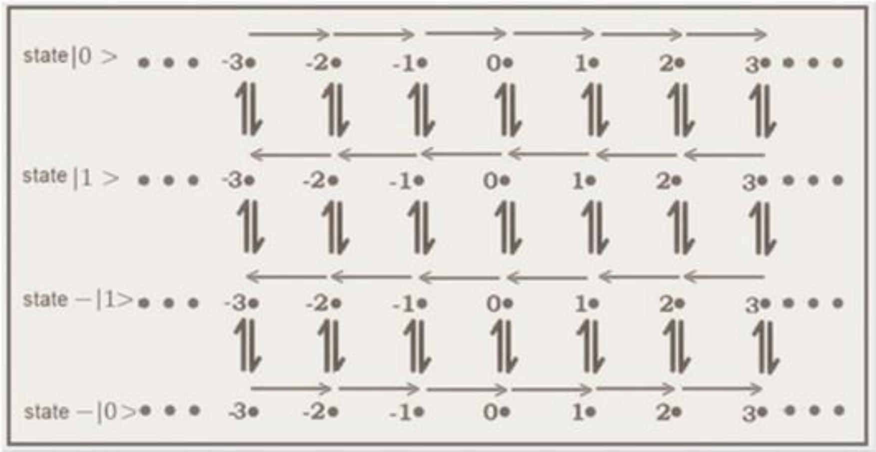


....

$S(\, I_d \otimes A) \rightarrow$

....

RW model



The main points of the RW model are as follows:

1.The shift operator is responsible for the movement in the horizontal direction (x-axis).

2.The transition matrix A is responsible for the movement in the vertical direction (y-axis).

3.The movements in each direction occur one after the other, rather than simultaneously.

3. Mathematical transformation between a Hadamard walk and a RW

	Hadamard walk	RW model
Hadamard operator	$y = I_d \otimes H$	$Y = I_d \otimes A$
Shift operator	$x = Right \otimes Zero + Left \otimes One$	$X = Right \otimes \hat{Z}ero + Left \otimes \hat{O}ne$
The whole process	$u = xy = Right \otimes ZeroH + Left \otimes OneH$	$U = XY = Right \otimes \hat{Z}eroA + Left \otimes \hat{O}neA$

And the equivalence between the Hadamard walk and the RW is proved by the following relation[1]:

$$|u^n (I_d \otimes B) P(0)| = \left(\sqrt{2}\right)^n |(I_d \otimes B) U^n P(0)| = \left(\sqrt{2}\right)^n |(I_d \otimes B) P(n)| \qquad (3)$$

Where n represents the step number,

$$Zero = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad One = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{Z}ero = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{O}ne = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The *right* and the *left* matrices are $d \times d$ zero matrices, except those entries that appear above and beneath the main diagonal, respectively, as follows:

$$Right = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad Left = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

4. From a RW Model → to a QRW

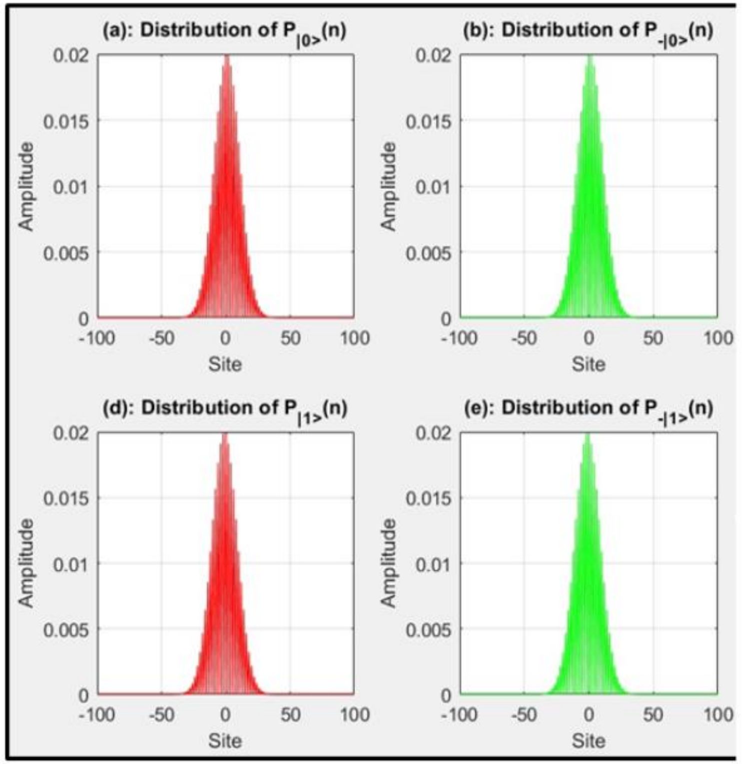
Thus, based on the RW model ,Eq. (3), the probability distributions of quantum states $|0>$ and $|1>$ after n steps in space are of the following magnitudes squared, respectively:

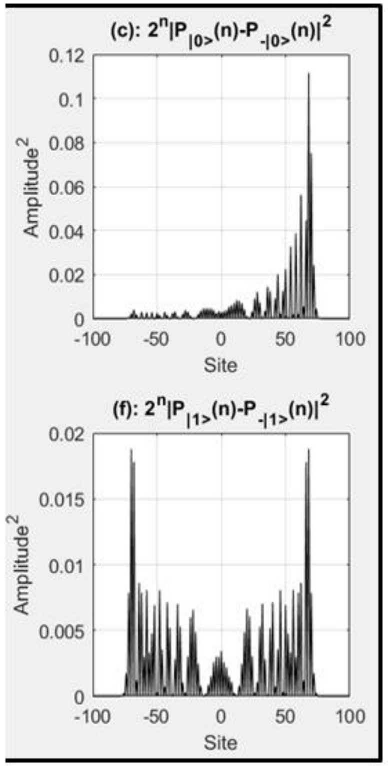
$$\left|\psi_{|0>}(n)\right|^2 = 2^n \left|P_{|0>}(n) - P_{-|0>}(n)\right|^2 \qquad (4)$$

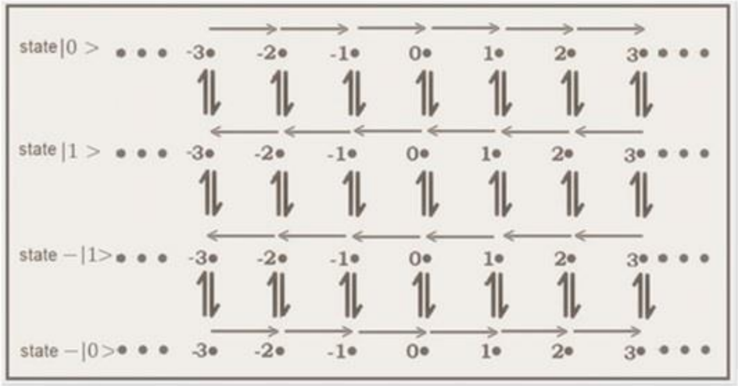
$$\left|\psi_{|1>}(n)\right|^2 = 2^n \left|P_{|1>}(n) - P_{-|1>}(n)\right|^2$$

The right figure presents the known results of a Hadamard walk starting at the origin with a $|0>$ quantum state after 100 steps using the probability distributions of the RW

Note the Gaussian distribution of the RW model before interference occurs





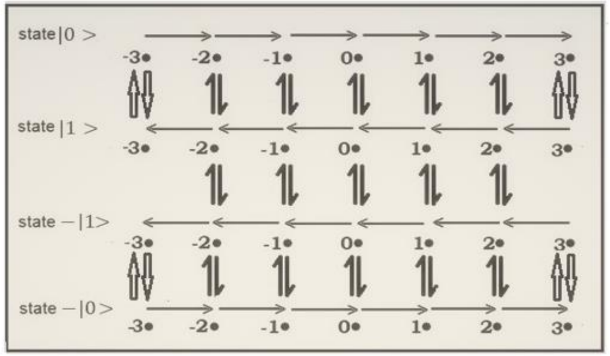


5. Inserting boundaries into the RW model

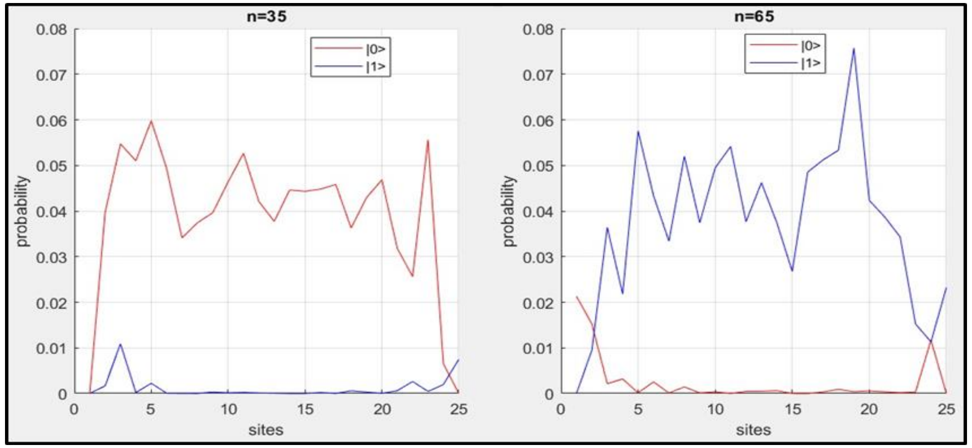
The following describes one possible reflecting barrier, which involves switching between the states $|0>, |1>$ and $-|0>, -|1>$ as presented by the matrice R_1 :

$$R_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The figure below describes a finite system with reflecting boundaries at locations -3 and 3, presented by R_1



The next figure describes the probability distribution of a QRW on a finite chain of 25 sites after 35 and 65 steps, calculated by a finite RW model, where each site (except the boundaries) initializes at $(|0> -|1>)/\sqrt{46}$. Note that the system initializes with equally distributed states, then switches mostly to state |0>, and after 65 steps mostly goes back to state |1>.



Summary

A new model that maps a quantum random walk to a particular case of a RW model was presented here for both finite and infinite cases. The RW walk model has completely different properties. For example, an infinite system has a stochastic matrix; namely, the system conserves the random walk population. Above all, this model shows that the dynamics between the RW model and a QRW are very similar, and it is the interference at the end of the process that makes the results of the QRW so different from a RW. [1] A.Bar-Haim arXiv:2006.11090 [quant-ph]